

The Signature Triality of Majorana-Weyl Spacetimes

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Abstract

Higher dimensional Majorana-Weyl spacetimes present space-time dualities which are induced by the $Spin(8)$ triality automorphisms. Different signature versions of theories such as 10-dimensional SYM's, superstrings, five-branes, F-theory, are shown to be interconnected via the S_3 permutation group. Bilinear and trilinear invariants under space-time triality are introduced and their possible relevance in building models possessing a space-versus-time exchange symmetry is discussed. Moreover the Cartan's "vector/chiral spinor/antichiral spinor" triality of $SO(8)$ and $SO(4,4)$ is analyzed in detail and explicit formulas are produced in a Majorana-Weyl basis. This paper is the extended version of hep-th/9907148.

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1 Introduction.

Physical theories formulated in different-than-usual spacetimes signatures have recently found increased attention. One of the reasons can be traced to the conjectured F -theory [1] which supposedly lives in $(2 + 10)$ dimensions [2]. The current interest in AdS theories motivated by the AdS/CFT correspondence furnishes another motivation. Two-time physics e.g. has started been explored by Bars and collaborators in a series of papers [3]. From another point of view we can also recall that a fundamental theory is expected to explain not only the spacetime dimensionality, but even its signature (see [4]). Quite recently Hull and Hull-Khuri [5] pointed out the existence of dualities relating different compactifications of theories formulated in different signatures. Such a result provides new insights to the whole question of spacetime signatures. In another context (see e.g. [6]) the existence of space-time dualities has also been remarked.

Majorana-Weyl spacetimes (i.e. those supporting Majorana-Weyl spinors) are at the very core of the present knowledge of the unification via supersymmetry, being at the basis of ten-dimensional superstrings, superYang-Mills and supergravity theories (and perhaps the already mentioned F -theory). A well-established feature of Majorana-Weyl spacetimes is that they are endorsed of a rich structure. A legitimate question that could be addressed is whether they are affected, and how, by space-time dualities. The answer is positive. Indeed all different Majorana-Weyl spacetimes which are possibly present in any given dimension are each-other related by duality transformations which are induced by the $Spin(8)$ triality automorphisms. The action of the triality automorphisms is quite non-trivial and has far richer consequences than the \mathbf{Z}_2 -duality (its most trivial representative) associated to the space-time $(s, t) \leftrightarrow (t, s)$ exchange discussed in [4]. It corresponds to S_3 , the six-element group of permutations of three letters, identified with the group of congruences of the triangle and generated by two reflections. The lowest dimension in which the triality action is non-trivial is 8 (not quite a coincidence), where the spacetimes $(8 + 0) - (4 + 4) - (0 + 8)$ are all interrelated. They correspond to the transverse coordinates of the $(9 + 1) - (5 + 5) - (1 + 9)$ spacetimes respectively, where the triality action can also be lifted. Triality relates as well the 12-dimensional Majorana-Weyl spacetimes $(10 + 2) - (6 + 6) - (2 + 10)$, i.e. the potentially interesting cases for the F -theory, and so on. Triality allows explaining the presence of points (read theory) in the brane-molecule table of ref. [4], corresponding to the different versions of e.g. superstrings, 11-dimensional supermembranes, fivebranes.

As a consequence of triality, supersymmetric theories formulated with Majorana-Weyl spinors in a given dimension but with different signatures, are all dually mapped one into another. A three-language dictionary is here furnished with the exact translations among the different versions of such supersymmetric theories. It should be stressed that, unlike [5], the dualities here discussed are already present for the *uncompactified* theories and in this respect look more fundamental.

The reason why the triality of the $d = 8$ dimension plays a role in dually relating higher dimensional spacetimes is a consequence of the fact that higher-dimensional Lorentz groups

admit the 8-dimensional Lorentz as a subgroup. This feature is neatly encoded in the representation properties of the higher dimensional Clifford Γ -matrices. In fact, due to this argument, it can be shown that not only different-signature Majorana-Weyl spacetimes are duality-related, but also that different-signatures odd-dimensional Majorana spacetimes are all interconnected via triality. This is true in particular for the 11-dimensional case relevant for the maximal supergravity and the M -theory.

Manifestations of triality are observed in different contexts. We will stress the fact that besides the original “Cartan” triality [7] exhibited by vectors, chiral and antichiral spinors (and therefore also denoted as “VCA”-triality in the following) in each one of the three Majorana-Weyl spacetimes of signature $(8+0)$, $(4+4)$, $(0+8)$ respectively, consequences of triality are found at the Clifford’s Γ matrices representation level. In $d = 8$ dimension this is seen by the fact that “Majorana type representations” for Γ -matrices, i.e. such that all the Γ ’s have a definite (anti-)symmetry property, only exist for the $4_S + 4_A$, $8_S + 0_A$, $0_S + 8_A$ cases.

Moreover it can be shown that the three Majorana-Weyl spacetimes of signatures $(4+4)$, $(8+0)$, $(0+8)$ are interrelated via the S_3 permutation group. We call this property the “signature-triality” or the “space-time triality”.

It is worth stressing the fact that the arising of the S_3 permutation group as a signature-duality group for Majorana-Weyl spacetimes in a given dimension *is not* a completely straightforward consequence of the existence of Majorana-Weyl spacetimes in three different signatures. Some extra-requirements have to be fulfilled in order this to be true. As an example we just mention that a necessary condition for the presence of S_3 requires that each given couple of the three different spacetimes must differ by an *even* number of signatures (in the text this point will be discussed in full detail); the flipping of an *odd* number of signatures, like in the Wick rotation from Minkowski to the Euclidean space, cannot be achieved with a \mathbf{Z}_2 group when spinors are involved.

In the present paper various aspects of the triality property will be rather extensively discussed. This is due to the objective relevance of the transverse 8-dimensional space of coordinates in the light-cone formulations of superstrings theories, branes, and the non-perturbative (M)atrix approach to M -theory. The eight dimensions are indeed the natural setting for the appearance of triality-related symmetries. Triality is of course a very well-known property which has been extensively investigated in the literature both for technical reasons (as an example it implies the equivalence of the NSR and GS formulation of superstrings, see e.g. [8]) as well in its more fundamental aspects. In this paper, besides discussing the signature triality, we express its action on the Majorana-type representations of the Clifford’s Γ matrices. Moreover, for completeness, we review and extend some of the original Cartan’s results on V-C-A triality.

He gave a concrete representation within a non-diagonal metric which, once diagonalized, shows the $(4+4)$ -signature. We point out that the triality generators he produced *do not* close the S_3 permutation group. However, an alternative presentation is available which implies the existence of S_3 . As a consequence the triality group \mathcal{G}_{Tr} defined in section 6 is given by the semidirect product of a linear subgroup \mathcal{G} (investigated also in [9]) of 24-dimensional matrices with such an S_3 permutation group. It is worth mentioning that in the formulas of paragraph 139 of Cartan’s book an obvious typographical mistake plus a sign error appear.

In the appendices 2-4 we produce the triality generators closing the S_3 group for each one of the three signatures $(4+4)$, $(8+0)$, $(0+8)$ which carry the triality structure. All formulas are here expressed in a Majorana-Weyl basis.

Furthermore, the construction of bilinear and trilinear invariants under the S_3 permutation group of the three Majorana-Weyl spacetimes is performed. They can be possibly used to formulate supersymmetric Majorana-Weyl theories in a manifestly triality-invariant form which presents an explicit symmetry under exchange of space and time coordinates.

The scheme of this work is as follows. In the next section we recall, following [10] and [11], the basic properties of Γ -matrices and Majorana conditions needed for our construction. Majorana-type representations are analyzed in section 3. We show there how to relate the Majorana-Weyl representations in $d > 8$ to the 8-dimensional Majorana-Weyl representations. In section 4 we introduce, for $d = 8$, the set of data necessary to define a supersymmetric Majorana-Weyl theory, i.e. the set of “words” of our three-languages dictionary. The Cartan’s [7] triality among vectors, chiral and antichiral spinors is presented in section 5. The main result is furnished in section 6, where spacetime triality is discussed. In section 7 the triality as a generator of a group of invariances is analyzed. In section 8 an application of triality is made. It is shown how to connect via triality some points (theories) presented in the Blencowe-Duff “brane-molecule scan” of reference [4] (further enlarged in the second paper referred in [4]). In the original paper only the presence of “mirror” theories connected by a \mathbf{Z}_2 space-versus-time exchange was “explained”. In the Conclusions we furnish some comments and point out some perspectives of future works. In the appendices, besides the already mentioned results, some useful construction concerning Γ matrices representations in 6 and 8 dimensions is given.

2 Preliminary results.

In this section the basic ingredients needed for our construction and the conventions employed will be introduced. More detailed information can be found in [10] and [11].

We denote as g_{mn} the flat (pseudo-)euclidean metric of a $(t + s)$ -spacetime. Time (space) directions in our conventions are associated to the $+$ (respectively $-$) sign.

The Γ ’s matrices are assumed to be unitary and without loss of generality a time-like Γ is normalized so that its square is $+1$. The three matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are the generators of the three conjugation operations (hermitian, complex conjugation and transposition respectively) on the Γ ’s. In particular

$$\mathcal{C}\Gamma^m\mathcal{C}^\dagger = \eta(-1)^{t+1}\Gamma^{mT} \quad (1)$$

where $\eta = \pm 1$ is a sign. In even-dimensional spacetimes it labels the two inequivalent choices of the charge conjugation matrix \mathcal{C} .

A relation exists, given by the formula

$$\mathcal{C} = \mathcal{B}^T \mathcal{A} \quad (2)$$

expressing anyone of the three matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}$ in terms of the two others.

Up to an inessential phase, \mathcal{A} is specified by the product of all the time-like Γ matrices.

An unitary transformations U applied on spinors act on $\Gamma^m, \mathcal{A}, \mathcal{B}, \mathcal{C}$ according to [11]

$$\begin{aligned} \Gamma^m &\mapsto U\Gamma^m U^\dagger \\ \mathcal{A} &\mapsto U\mathcal{A}U^\dagger \\ \mathcal{B} &\mapsto U^*\mathcal{B}U^\dagger \\ \mathcal{C} &\mapsto U^*\mathcal{C}U^\dagger \end{aligned} \quad (3)$$

A Majorana representation for the Γ 's can be defined as the one in which \mathcal{B} is set equal to the identity. Spinors can be assumed real in this case.

In even dimensions we can also introduce the notion of Weyl representation, i.e. when the “generalized Γ^5 matrix” is symmetric and block diagonal and with no loss of generality can be assumed to be the direct sum of the two equal-size blocks $\mathbf{1} \oplus (-\mathbf{1})$. The compatibility of both Majorana and Weyl conditions constraints the spacetime $(t + s)$ to satisfy

$$s - t = 0 \mod 8, \text{ for both values } \eta = \pm 1 \quad (4)$$

In even dimensions Majorana representations, but not of Weyl type, are also found for

$$\begin{aligned} s - t &= 2 \mod 8 \text{ for } \eta = -1; \\ s - t &= 6 \mod 8 \text{ for } \eta = +1. \end{aligned} \quad (5)$$

In odd dimensions Weyl spinors cannot be defined, while Majorana spinors exist for

$$s - t = 1, 7 \mod 8 \quad (6)$$

For $d < 8$ the only spacetimes supporting Majorana-Weyl spinors have signatures $(n + n)$. At $d = 8$ a new feature arises, Majorana-Weyl spinors can be found for different signatures.

Making explicit the relation between theories formulated in such different signatures is the main content of this paper.

3 Majorana-type representations.

It is convenient to introduce the notion of Majorana-type representation (or shortly MTR) of the Clifford's Γ matrices. It can be defined as a representation such that all the Γ 's have a definite symmetry, being either symmetric or antisymmetric. In d dimensions a MTR with p symmetric and q antisymmetric Γ 's ($p + q = d$) will be denoted as $(p_S + q_A)$ in the following.

When specialized to such representations the \mathcal{C} charge-conjugation matrix introduced in the previous section is given by either the product of all the symmetric Γ matrices, denoted as \mathcal{C}_S , or all the antisymmetric ones (\mathcal{C}_A)

$$\begin{aligned} \mathcal{C}_S &= \prod_{i=1, \dots, p} \Gamma^i_S \\ \mathcal{C}_A &= \prod_{i=1, \dots, q} \Gamma^i_A \end{aligned} \quad (7)$$

Please notice that the index S, A is not referred to the (anti-)symmetry property of the matrices $\mathcal{C}_{S,A}$ themselves.

In even dimensions $\mathcal{C}_S, \mathcal{C}_A$ correspond to opposite values of η in (1), while in odd dimensions, up to an inessential phase factor, the two definitions for the charge-conjugation matrix collapse into a single matrix. This is in agreement with the property that in odd dimensions, up to unitary conjugation, the \mathcal{C} -matrix is uniquely defined. The convenience of using MTR's to discuss Wick rotations to and from the Euclidean space has been advocated in [11].

It can be easily recognized that a Majorana representation for Clifford's Γ matrices in a given signature spacetime implies the Γ 's belong to a MTR. Conversely, given a MTR with p_S, q_A (anti-)symmetric Γ matrices, two spacetimes exist ($t = p, s = q$ with the choice $\mathcal{C} \equiv \mathcal{C}_S$,

and respectively $t = q$, $s = p$ for $\mathcal{C} \equiv \mathcal{C}_A$) such that the representation is Majorana (i.e. $\mathcal{B} = \mathbf{1}$). The admissible couples of (p_S, q_A) values for a MTR can be immediately read from the Majorana tables given above (4), (5) and (6). The construction is such that \mathcal{C} must correspond to the correct value of η appearing in the tables.

The list of all possible MTR's in any given dimension is therefore easily computed. In order to furnish an example we mention that in $d = 6$ there exists a MTR (not of Weyl kind) with 6 anticommuting Γ matrices plus an anticommuting Γ^7 ($0_S + 6_A, \Gamma^7_A$). It provides a Majorana basis for an Euclidean 6-dimensional space. A concrete realization of such a representation is presented in appendix 1¹.

For completeness we present a list of all MTR's up to $d = 18$. We obtain

d	W · R		NW · R
2 + 1	$(1_S + 1_A + \Gamma^3_S)$	\leftrightarrow	$(2_S + 0_A + \Gamma^3_A)$
4 + 1	$(2_S + 2_A + \Gamma^5_S)$	\leftrightarrow	$(3_S + 1_A + \Gamma^5_A)$
6 + 1	$(3_S + 3_A + \Gamma^7_S)$	\leftrightarrow	$(4_S + 2_A + \Gamma^7_A)$
6 + 1			$(0_S + 6_A + \Gamma^7_A)$
8 + 1	$(8_S + 0_A + \Gamma^9_S)$		
8 + 1	$(4_S + 4_A + \Gamma^9_S)$	\leftrightarrow	$(5_S + 3_A + \Gamma^9_A)$
8 + 1	$(0_S + 8_A + \Gamma^9_S)$	\leftrightarrow	$(1_S + 7_A + \Gamma^9_A)$
10 + 1	$(9_S + 1_A + \Gamma^{11}_S)$	\leftrightarrow	$(10_S + 0_A + \Gamma^{11}_A)$
10 + 1	$(5_S + 5_A + \Gamma^{11}_S)$	\leftrightarrow	$(6_S + 4_A + \Gamma^{11}_A)$
10 + 1	$(1_S + 9_A + \Gamma^{11}_S)$	\leftrightarrow	$(2_S + 8_A + \Gamma^{11}_A)$
12 + 1	$(10_S + 2_A + \Gamma^{13}_S)$	\leftrightarrow	$(11_S + 1_A + \Gamma^{13}_A)$
12 + 1	$(6_S + 6_A + \Gamma^{13}_S)$	\leftrightarrow	$(7_S + 5_A + \Gamma^{13}_A)$
12 + 1	$(2_S + 10_A + \Gamma^{13}_S)$	\leftrightarrow	$(3_S + 9_A + \Gamma^{13}_A)$
14 + 1	$(11_S + 3_A + \Gamma^{15}_S)$	\leftrightarrow	$(12_S + 2_A + \Gamma^{15}_A)$
14 + 1	$(7_S + 7_A + \Gamma^{15}_S)$	\leftrightarrow	$(8_S + 6_A + \Gamma^{15}_A)$
14 + 1	$(3_S + 11_A + \Gamma^{15}_S)$	\leftrightarrow	$(4_S + 10_A + \Gamma^{15}_A)$
14 + 1			$(0_S + 14_A + \Gamma^{15}_A)$
16 + 1	$(16_S + 0_A + \Gamma^{17}_S)$		
16 + 1	$(12_S + 4_A + \Gamma^{17}_S)$	\leftrightarrow	$(13_S + 3_A + \Gamma^{17}_A)$
16 + 1	$(8_S + 8_A + \Gamma^{17}_S)$	\leftrightarrow	$(9_S + 7_A + \Gamma^{17}_A)$
16 + 1	$(4_S + 12_A + \Gamma^{17}_S)$	\leftrightarrow	$(5_S + 11_A + \Gamma^{17}_A)$
16 + 1	$(0_S + 16_A + \Gamma^{17}_S)$	\leftrightarrow	$(1_S + 15_A + \Gamma^{17}_A)$
18 + 1	$(17_S + 1_A + \Gamma^{19}_S)$	\leftrightarrow	$(18_S + 0_A + \Gamma^{19}_A)$
18 + 1	$(13_S + 5_A + \Gamma^{19}_S)$	\leftrightarrow	$(14_S + 4_A + \Gamma^{19}_A)$
18 + 1	$(9_S + 9_A + \Gamma^{19}_S)$	\leftrightarrow	$(10_S + 8_A + \Gamma^{19}_A)$
18 + 1	$(5_S + 13_A + \Gamma^{19}_S)$	\leftrightarrow	$(6_S + 12_A + \Gamma^{19}_A)$
18 + 1	$(1_S + 17_A + \Gamma^{19}_S)$	\leftrightarrow	$(2_S + 16_A + \Gamma^{19}_A)$

Some comments are in order. In the second column we listed all MTR's of Weyl type, in the last one the non-Weyl representations. An arrow connects two given Weyl and non-Weyl

¹It is worth mentioning that such a representation can be written in terms of the octonionic structure constants (see ref. [12]). In the present work our results have been obtained making no explicit reference to octonions; investigating the connection with such a division algebra is outside the scope of this work.

representations which are intertwined by an exchange of the “generalized Γ^5 ” matrix with any other Γ -matrix of opposite symmetry. Up to $d = 18$ the only true genuine even-dimensional non-Weyl Majorana representations not having such a Weyl counterpart are the above-mentioned 6-dimensional $(0_S + 6_A)$ representation and the 14-dimensional $(0_S + 14_A)$, both appearing in the table. Their difference in dimensionality ($= 8$) is of course a consequence of the famous *mod* 8 property of the Γ matrices, see e.g. [13].

Different MTR’s belong to different classes under similarity transformations of the Γ ’s representations. Indeed in, let’s say, an euclidean (all + signs) space, the index

$$I = \text{tr}(\Gamma^m \cdot \Gamma_m^T) = (p_S - q_A) \cdot \text{tr} \mathbf{1} \quad (9)$$

takes different values for different MTR’s and is by construction invariant under the transformation $\Gamma \mapsto O\Gamma O^T$ realized by orthogonal matrices O .

Up to $d = 8$ (excluded) there exists a unique similarity class of MTR’s of Weyl type, so that Majorana-Weyl spinors can be defined in $(n + n)$ space-times only. A new feature arises starting from $d \geq 8$, Weyl and Majorana-type representations are compatible for different similarity classes. In $d = 8$ the three solutions $(8_S + 0_A)$, $(4_S + 4_A)$, $(0_S + 8_A)$, associated with the corresponding Majorana-Weyl spacetimes, are found.

An efficient tool to produce and analyze Weyl representations in any higher dimensions is furnished by the following algorithm [14] and [11]. It provides a recursive procedure to construct d -dimensional Weyl representations in terms of any given couple of r, s lower-dimensional Γ -matrices representations, where the even integers d, r, s are constrained to satisfy

$$d = r + s + 2 \quad (10)$$

Moreover, if the r, s -dimensional representations are of Majorana-type, the d -dimensional one is Majorana-Weyl.

The algorithm can be expressed through the formula

$$\begin{aligned} \Gamma_d^{i=1, \dots, s+1} &= \sigma_x \otimes \mathbf{1}_L \otimes \Gamma_s^{i=1, \dots, s+1} \\ \Gamma_d^{s+1+j=s+2, \dots, d} &= \sigma_y \otimes \Gamma_r^{j=1, \dots, r+1} \otimes \mathbf{1}_R \end{aligned} \quad (11)$$

where $\mathbf{1}_{L,R}$ are the unit-matrices in the respective spaces, while $\sigma_x = e_{12} + e_{21}$ and $\sigma_y = -ie_{12} + ie_{21}$ are the 2-dimensional Pauli matrices. Γ_r^{r+1} corresponds to the “generalized Γ^5 -matrix” in $r + 1$ dimensions. In the above formula the values $r, s = 0$ are allowed. The corresponding Γ_0^1 is just 1.

By iteratively applying the above algorithm starting from 1 (that is $r, s = 0$), we obtain as a first step the 2-dimensional Pauli matrices and next, from $r = 0, s = 2$, the 4-dimensional MW representation as a second step. The 6-dimensional MW representation $(3_S + 3_A)$ is obtained as a further step. It can be produced either from $r = 0, s = 4$, or $r = 2, s = 2$. The non-Weyl $(0_S + 6_A)$ representation is constructed with the method explained in appendix 1.

The higher-dimensional MW representations as well are obtained via the algorithm. An

explicit way of constructing them is, e.g., in accordance with the table

$r = 6$	$(0_S + 6_A)$	$s = 0$	(1)	\mapsto	$(8_S + 0_A)$
$r = 6$	$(3_S + 3_A)$	$s = 0$	(1)	\mapsto	$(4_S + 4_A)$
$r = 0$	(1)	$s = 0$	$(0_S + 6_A)$	\mapsto	$(0_S + 8_A)$
$r = 0$	(1)	$s = 8$	$(8_S + 0_A)$	\mapsto	$(9_S + 1_A)$
$r = 8$	$(4_S + 4_A)$	$s = 0$	(1)	\mapsto	$(5_S + 5_A)$
$r = 8$	$(8_S + 0_A)$	$s = 0$	(1)	\mapsto	$(1_S + 9_A)$
$r = 0$	(1)	$s = 10$	$(9_S + 1_A)$	\mapsto	$(10_S + 2_A)$
$r = 0$	(1)	$s = 10$	$(5_S + 5_A)$	\mapsto	$(6_S + 6_A)$
$r = 0$	(1)	$s = 10$	$(1_S + 9_A)$	\mapsto	$(2_S + 10_A)$
$r = 0$	(1)	$s = 12$	$(10_S + 2_A)$	\mapsto	$(11_S + 3_A)$
$r = 0$	(1)	$s = 12$	$(6_S + 6_A)$	\mapsto	$(7_S + 7_A)$
$r = 0$	(1)	$s = 12$	$(2_S + 10_A)$	\mapsto	$(3_S + 11_A)$
$r = 6$	$(0_S + 6_A)$	$s = 8$	$(8_S + 0_A)$	\mapsto	$(16_S + 0_A)$
$r = 0$	(1)	$s = 14$	$(11_S + 3_A)$	\mapsto	$(12_S + 4_A)$
$r = 0$	(1)	$s = 14$	$(7_S + 7_A)$	\mapsto	$(8_S + 8_A)$
$r = 0$	(1)	$s = 14$	$(3_S + 11_A)$	\mapsto	$(4_S + 12_A)$
$r = 8$	$(8_S + 0_A)$	$s = 6$	$(0_S + 6_A)$	\mapsto	$(0_S + 16_A)$
$r = 0$	(1)	$s = 16$	$(16_S + 0_A)$	\mapsto	$(17_S + 1_A)$
$r = 0$	(1)	$s = 16$	$(12_S + 4_A)$	\mapsto	$(13_S + 5_A)$
$r = 0$	(1)	$s = 16$	$(8_S + 8_A)$	\mapsto	$(9_S + 9_A)$
$r = 0$	(1)	$s = 16$	$(4_S + 12_A)$	\mapsto	$(5_S + 13_A)$
$r = 0$	(1)	$s = 16$	$(0_S + 16_A)$	\mapsto	$(1_S + 17_A)$

(12)

Notice that in order to explicitly construct all MW-representations up to $d = 18$ with the help of formula (11), the only extra-knowledge of the $(0_S + 6_A)$ non-Weyl Majorana is required.

In the following we will refer to “space-time triality” as the 8-dimensional property that the Majorana-Weyl condition is satisfied in three different signatures and will show its connection to the usual triality property of the 8 dimensions, as well as the S_3 permutation group.

Due to the “lifting” formula (11), this feature is extended to higher-dimensional MW-spacetimes. Solutions in three different signatures arise as well in $d = 10$, $d = 12$ and $d = 14$. Such solutions are a direct consequence of the embedding of the eight-dimensional Lorentz algebra into higher dimensions. In dimensions higher than 8 the property that Majorana-Weyl conditions (or simply Majorana conditions in odd dimensions) are consistent in different signatures can therefore be regarded as a “derived” property which is fundamentally rooted in the 8-dimensions. This argument holds even in the case (for $d \geq 16$), where solutions to the MW-constraints in more than three different signatures are obtained. As an example the $d = 18$ case can be produced with the help of (11) for the values $r = s = 8$. Therefore the 5 different 18-dimensional Majorana-Weyl representations are obtained from tensoring two 8-dimensional Majorana-Weyl representations.

On a physical ground and not just for purely mathematical purposes, at the present state of the art we do not need getting involved into such complications since the most promising dimensions where the ultimate candidate theories for unification are expected to live correspond to $d = 10, 11, 12$.

As a final comment in this section we emphasize once more that all data needed to define theories in such dimensions can be recovered, through a set of reconstruction formulas based on (11), from the 8-dimensional data. In particular all “space-time trialities” in $d > 8$ are encoded in the 8-dimensional “space-time triality”. For this reason in the following we can concentrate ourselves in investigating the 8-dimensional case.

4 The set of data for Majorana-Weyl supersymmetric theories.

In this section we present the set of data needed to specify a supersymmetric theory involving Majorana-Weyl spinors. The most suitable basis one can use in this case is the Majorana-Weyl basis previously discussed, where all spinors are either real or imaginary. In such a representation the following set of data underlines any given theory:

- i)** the vector fields (or, in the string/brane picture, the bosonic coordinates of the target x_m), specified by a vector index denoted by m ;
- ii)** the spinor fields (or, in the string/brane picture, the fermionic coordinates of the target $\psi_a, \chi_{\dot{a}}$), specified by chiral and antichiral indices a, \dot{a} respectively;
- iii)** the diagonal (pseudo-)orthogonal spacetime metric $(g^{-1})^{mn}, g_{mn}$ which we will assume to be flat;
- iv)** the \mathcal{A} matrix introduced in section 2, used to define barred spinors, coinciding with the Γ^0 -matrix in the Minkowski case; in a MWR is decomposed in an equal-size block diagonal form such as $\mathcal{A} = A \oplus \tilde{A}$, with structure of indices $(A)_a{}^b$ and $(\tilde{A})_{\dot{a}}{}^{\dot{b}}$ respectively;
- v)** the charge-conjugation matrix \mathcal{C} which also appears in an equal-size block diagonal form $\mathcal{C} = C^{-1} \oplus \tilde{C}^{-1}$. It is invariant under bispinorial transformations and it can be promoted to be a metric in the space of chiral (and respectively antichiral) spinors, used to raise and lower spinorial indices. Indeed we can set $(C^{-1})^{ab}, (C)_{ab}$, and $(\tilde{C}^{-1})^{\dot{a}\dot{b}}, (\tilde{C})_{\dot{a}\dot{b}}$;
- vi)** the Γ -matrices, which are decomposed in equal-size blocks as in (43), where the σ^m ’s are upper-right blocks and the $\tilde{\sigma}^m$ ’s lower-left blocks having structure of indices $(\sigma^m)_a{}^b$ and $(\tilde{\sigma}^m)_{\dot{a}}{}^{\dot{b}}$ respectively;
- vii)** the $\eta = \pm 1$ sign, labeling the two inequivalent choices for \mathcal{C} .

We recall that by definition the \mathcal{B} matrix is automatically set to be the identity ($\mathcal{B} = \mathbf{1}$) in a Majorana-Weyl representation.

The above structures are common in any theory involving Majorana-Weyl spinors. In the following we will furnish a dictionary relating Majorana-Weyl spacetimes with the same dimensionality, but with different signatures. The structures **i)-vii)** will be related via triality transformations which close the S_3 permutation group. They constitute the “words” in a three-language dictionary. According to the discussion in the previous section without loss of generality we can limit ourselves in analyzing the $d = 8$ -dimensional case. In this particular dimension the three indices, vector (m), chiral (a) and antichiral (\dot{a}) take values $m, a, \dot{a} \in \{1, \dots, 8\}$.

We mention that in the $(4+4)$ -signature the $(4_S + 4_A)$ -representation of the Γ -matrices has to be employed for both values of η in order to provide a Majorana-Weyl basis. In the $(t = 8, s = 0)$ signature the $(8_S + 0_A)$ -representation offers a MW basis for $\eta = +1$, while the $(0_S + 8_A)$ offers it for $\eta = -1$. The converse is true in the $(t = 0, s = 8)$ -signature.

The three 8-dimensional MW-type representations are explicitly constructed in the appendices 2-4, together with their charge-conjugation matrices \mathcal{C} 's.

5 The Cartan's V-C-A triality.

In this section we review the basic features of the Cartan's triality involving vectors, chiral and antichiral spinors of $SO(8)$ and $SO(4,4)$.

The fundamental reason behind triality is the peculiar property of the D_4 Lie algebra, the only one admitting a group of symmetry for the corresponding Dynkin diagram other than the identity or \mathbf{Z}_2 . Its group of symmetry is the six-elements non-abelian group of permutations S_3 which, as well-known, corresponds to the outer automorphisms ($Out \equiv Aut/Int$) of D_4 .

The groups $SO(8)$ and $SO(4,4)$ are obtained by exponentiating different real forms of the D_4 Lie algebra.

For such groups and the corresponding metrics, the euclidean one with either all + or all - signs and the pseudoeuclidean metric (+ + + + - - - -), the spinor representations are Majorana-Weyl and satisfy the properties discussed in the previous section. In particular chiral and antichiral spinors can be consistently defined.

A unique and "miraculous" feature of the above spacetimes, not shared by any other case, consists in the fact that vectors, chiral and antichiral spinors (in short V-C-A's) have the same dimensionality, being all eight-components.

Transformations exchanging V-C-A's are found and, as we discuss later, they can all be identified. Such a property can be visualized with the following triangle diagram whose vertices are the interrelated V-C-A's:

$$\begin{array}{ccc} & V & \\ \circ & & \circ \\ C & \circ & A \end{array} \quad (13)$$

Vectors (V_m), chiral (ψ_a) and antichiral ($\chi_{\dot{a}}$) spinors can be conveniently arranged into a single 24-component "triality vector" T

$$T_M = \begin{pmatrix} V_m \\ \psi_a \\ \chi_{\dot{a}} \end{pmatrix} \quad (14)$$

whose Lorentz-transformation properties are given by

$$T \mapsto T' = e^{\frac{1}{2}\omega_{mn}\Sigma^{mn}} \cdot T \quad (15)$$

where

$$\Sigma^{mn} = \Sigma_V^{mn} \oplus \Sigma_C^{mn} \oplus \Sigma_A^{mn} \quad (16)$$

while $(\Sigma_{V,C,A})^{mn}$ are the Lorentz-generators for vectors, chiral and antichiral spinors respectively:

$$\begin{aligned} (\Sigma_V^{mn})_k^l &= \delta_k^m (g^{-1})^{nl} - (g^{-1})^{ml} \delta_k^n \\ \Sigma_C^{mn} &= \frac{1}{4}(\sigma^m \tilde{\sigma}^n - \sigma^n \tilde{\sigma}^m) \\ \Sigma_A^{mn} &= \frac{1}{4}(\tilde{\sigma}^m \sigma^n - \tilde{\sigma}^n \sigma^m) \end{aligned} \quad (17)$$

As far as the Lorentz transformation properties alone are concerned, there is no need to discuss (and even introduce) the character, commuting or anticommuting, of spinors. However, the triality admits to be interpreted as an invariance property which allows to introduce the \mathcal{G}_{Tr} triality group of symmetry (to be defined below). For such interpretation we need to specify whether the spinors are assumed commuting or anticommuting. Following Cartan we discuss here the case of commuting spinors. We will comment the modifications occurring when anticommuting spinors are taken into account².

For both values of $\eta = \pm 1$, the following bilinear Lorentz-invariants can be introduced

$$\begin{aligned}\mathcal{B}_V &= V^T{}_m (g^{-1})^{mn} V_n \\ \mathcal{B}_C &= \psi^T C^{-1} \psi \\ \mathcal{B}_A &= \chi^T \tilde{C}^{-1} \chi\end{aligned}\tag{18}$$

(we use the conventions discussed in the previous section), while the trilinear Lorentz-invariant

$$\mathcal{T} = \Psi^T \mathcal{C} \Gamma^m \Psi \cdot V_m = 2(\psi^T C^{-1} \sigma^m \chi \cdot V_m), \quad \Psi \equiv \begin{pmatrix} \psi_a \\ \chi_{\dot{a}} \end{pmatrix}\tag{19}$$

is non-vanishing for $\eta = -1$. For anticommuting spinors the bilinears \mathcal{B}_C , \mathcal{B}_A are identically vanishing, while \mathcal{T} is non-vanishing for $\eta = +1$.

The group of invariances \mathcal{G} is introduced as the group of linear homogeneous transformations acting on the $8 \times 3 = 24$ dimensional space of “triality-vectors” leaving invariant, separately, \mathcal{B}_V , \mathcal{B}_C , \mathcal{B}_A and \mathcal{T} .

The group of triality \mathcal{G}_{Tr} is defined by relaxing one condition, as the group of 24-dimensional homogeneous linear transformations leaving invariant \mathcal{T} and the total bilinear \mathcal{B}_{Sum}

$$\mathcal{B}_{Sum} = \mathcal{B}_V + \mathcal{B}_C + \mathcal{B}_A\tag{20}$$

It can be proven that \mathcal{G}_{Tr} is given by the semidirect product of \mathcal{G} and the finite group S_3

$$\mathcal{G}_{Tr} = \mathcal{G} \otimes_S S_3\tag{21}$$

This result directly follows from the Cartan’s approach, even if it is not explicitly stated in the Cartan’s book³. The transformations in S_3 are obtained from the generators \mathcal{P} , \mathcal{R} whose action, symbolically, is given by

$$\begin{aligned}\mathcal{P} &: V \leftrightarrow V, \quad C \leftrightarrow A \\ \mathcal{R} &: V \leftrightarrow C, \quad A \leftrightarrow A\end{aligned}\tag{22}$$

As a consequence \mathcal{P} , \mathcal{R} can be decomposed into eight-dimensional blocks matrices according to

$$\mathcal{P} = \begin{pmatrix} P_1 & & \\ & P_2 & \\ & & P_3 \end{pmatrix}, \quad \mathcal{R} = \begin{pmatrix} & R_1 & \\ R_2 & & \\ & & R_3 \end{pmatrix}\tag{23}$$

²Commuting spinors too are relevant in the physical literature, not only as supersymmetric ghosts in a BRST quantization scheme; they also appear, e.g., in the super-twistors quantization approach, see [15]. The fundamental reference on twistors is [16].

³He introduced the analogs of the \mathcal{P} , \mathcal{R} transformations of formula (22) which however, in his case, do not satisfy the relations (24) and cannot therefore be taken as the generators of the S_3 group.

\mathcal{P}, \mathcal{R} can be carefully chosen in such a way to satisfy the relations

$$\mathcal{P}^2 = \mathcal{R}^2 = \mathbf{1}, \quad (\mathcal{P}\mathcal{R})^3 = \mathbf{1} \quad (24)$$

showing their nature of S_3 generators.

There is an arbitrariness in the choice of \mathcal{P}, \mathcal{R} since any couple of gauge-transformed generators $\hat{\mathcal{P}} = g \cdot \mathcal{P} \cdot g^{-1}$, $\hat{\mathcal{R}} = g \cdot \mathcal{R} \cdot g^{-1}$, with $g \in \mathcal{G}_{Tr}$, satisfy the same properties and can be equally taken for generating S_3 . It should be mentioned that, under a Lorentz transformation realized by $e^{\omega \cdot \Sigma}$, the generators \mathcal{P} (\mathcal{R}) are mapped into $\mathcal{P}' = e^{\omega \cdot \Sigma} \mathcal{P} e^{\omega \cdot \Sigma^{-1}}$ (and respectively $\mathcal{R}' = e^{\omega \cdot \Sigma} \mathcal{R} e^{\omega \cdot \Sigma^{-1}}$). However, since we are free to introduce a gauge-compensating transformation, we can make invariant the choice of \mathcal{P}, \mathcal{R} in any Lorentzian system of reference. As a consequence the constants \mathcal{P}, \mathcal{R} matrices can be introduced in manifestly Lorentz-invariant actions if needed.

It is not easy, due to the intertwined nature of their relations (24), to determine a concrete realization for \mathcal{P}, \mathcal{R} . We arrived at it with a trial-and-error procedure. The final result is furnished in the appendices 2-4 for each one of the three possible metrics.

If we disregard \mathcal{B}_{Sum} and just demand the invariance of \mathcal{T} , in this case both \mathcal{P}, \mathcal{R} can always be assumed real-valued. This is no longer true when the invariance of \mathcal{B}_{Sum} is required, due to the fact that the three metrics on V-C-A's can differ by an overall $-$ sign. This case is easily managed by noticing that the transformations $P_2 \mapsto iP_2$, $P_3 \mapsto -iP_3$, P_1 unchanged (and $R_1 \mapsto iR_1$, $R_2 \mapsto -iR_2$, R_3 unchanged) solves the problem without altering neither the (24) relations nor the \mathcal{T} -invariance requirement.

We finally comment that vectors, chiral and antichiral spinors can be identified under triality. If a basis in these three different spaces has already be chosen, the most natural way of identifying them ($V \equiv X_1 \cdot C$, $V \equiv Y_1 \cdot A$) make use of the eight-dimensional invertible operators X_1, Y_1 entering

$$\mathcal{P} \cdot \mathcal{R} = \begin{pmatrix} & X_1 & \\ & & X_2 \\ X_3 & & \end{pmatrix}, \quad \mathcal{R} \cdot \mathcal{P} = \begin{pmatrix} & & Y_1 \\ Y_2 & & \\ & Y_3 & \end{pmatrix} \quad (25)$$

The reason to use them is because X_1, Y_1 map into vectors respectively chiral and antichiral spinors with transformations which correspond to *even* permutations of V,C,A.

6 The signature triality.

In this section we discuss other consequences of the S_3 automorphisms of D_4 . Besides being responsible for the Cartan's V-C-A triality in fact, triality properties are associated with other structures. For purpose of clarity it will be convenient to represent them symbolically with triangle diagrams as the one shown in (13).

An extra-consequence of triality appears at the level of Majorana-Weyl representations for Clifford's Γ -matrices, see section 3. The three different eight-dimensional representations can in fact be placed into the diagram

$$\begin{array}{ccc} & (4_S + 4_A) & \\ \circ & & \circ \\ (8_S + 0_A) & \circ & (0_S + 8_A) \end{array} \quad (26)$$

which exhibits the triality operating at the level of the Γ -matrices.

We have recalled that such MW-representations are associated with the space-time signature, and therefore triality can also be regarded as operating on space-times according to

$$\left(\begin{array}{ccc} & (5+5) & \\ & \circ & \circ \\ (9+1) & & (1+9) \end{array} \right) \longrightarrow \left(\begin{array}{ccc} & (4+4) & \\ & \circ & \circ \\ (8+0) & & (0+8) \end{array} \right) \quad (27)$$

The arrow has been inserted to recall that such triality can be lifted to higher dimensions or, conversely, that the 8-dimensional spacetimes arise as transverse-coordinates spaces in physical theories.

The triality operating at the level of spacetime signatures is the one visualized by the previous diagram and will be described in this section.

Cartan's V-C-A triality and signature triality can also be combined and symbolically represented by a sort of fractal-like double-triality diagram as follows

$$\begin{array}{c} V \\ \cdot \quad (4+4) \quad \cdot \\ C \quad \cdot \quad A \\ \circ \qquad \qquad \qquad \circ \end{array} \quad (28)$$

$$\begin{array}{ccc} V & & V \\ \cdot \quad (8+0) \quad \cdot & \circ & \cdot \quad (0+8) \quad \cdot \\ C \quad \cdot \quad A & & C \quad \cdot \quad A \end{array}$$

The bigger triangle illustrates the signature triality, while the smaller triangles visualize the trialities for vectors, chiral and antichiral spinors living in each space-time.

We give now the explicit expression of the duality transformations relating theories formulated in the three above spacetimes or, in other words, the “translation rules” for the set of data discussed in section 4.

To be definite we discuss the $\eta = -1$ case (we recall that the η -sign has been introduced in section 2); the modifications to be introduced in the $\eta = +1$ case are immediate.

Since we are working in a Majorana-Weyl basis it is always true that $\mathcal{B} = \mathbf{1}$. The data defining our theories are therefore specified by the metric in the vectors' space g^{-1} , as well as the charge-conjugation matrix \mathcal{C} which contains both the metric for chiral (C^{-1}) and antichiral (\tilde{C}^{-1}) spinors. In a Majorana-Weyl basis the \mathcal{A} matrix is identified with \mathcal{C} via the equation (2).

It is convenient to collectively denote as g_{\star}^{-1} (where $\star \equiv V, C, A$) the three metrics associated to, respectively, vectors and chiral and antichiral spinors in the $(4+4)$ spacetime. The analogous metrics when associated to the $(8+0)$ spacetime will be denoted with a tilde (\tilde{g}_{\star}^{-1}), while the hat will denote the three metrics associated to the $(0+8)$ spacetime (\hat{g}_{\star}^{-1}).

Working in the $\eta = -1$ case with the representations given in appendices 2 – 4 we have

$$g_V^{-1} = \mathbf{1}_4 \oplus -\mathbf{1}_4, \quad g_C^{-1} = \mathbf{1}_4 \oplus -\mathbf{1}_4, \quad g_A^{-1} = \mathbf{1}_4 \oplus -\mathbf{1}_4;$$

$$\begin{aligned}\tilde{g}_V^{-1} &= \mathbf{1}_8, & \tilde{g}_C^{-1} &= \mathbf{1}_8, & \tilde{g}_A^{-1} &= -\mathbf{1}_8; \\ \hat{g}_V^{-1} &= -\mathbf{1}_8, & \hat{g}_C^{-1} &= \mathbf{1}_8, & \hat{g}_A^{-1} &= \mathbf{1}_8;\end{aligned}\tag{29}$$

The duality transformations can therefore be expressed by similarity transformations realized by non-orthogonal bridge matrices connecting the different metrics. We can introduce indeed the eight-dimensional bridge matrices $\tilde{K}_\star, \hat{K}_\star$ such that

$$\begin{aligned}\tilde{g}_\star^{-1} &= \tilde{K}_\star \cdot g_\star^{-1} \cdot \tilde{K}_\star^T \\ \hat{g}_\star^{-1} &= \hat{K}_\star \cdot g_\star^{-1} \cdot \hat{K}_\star^T\end{aligned}\tag{30}$$

(as before “ \star ” assumes the values V, C, A).

Vectors, chiral and antichiral spinors (V-C-A’s, collectively denoted as φ_\star) in the $(4+4)$ spacetime are transformed into the $(8+0)$ -signature V-C-A’s $\tilde{\varphi}_\star$ according to

$$\tilde{\varphi}_\star \equiv (\tilde{K}_\star^T)^{-1} \cdot \varphi_\star\tag{31}$$

An analogous transformation maps them into the $(0+8)$ -signature V-C-A’s $\hat{\varphi}_\star$.

The 16-dimensional matrices \tilde{H}, \hat{H} are constructed with the help of \tilde{K}_C, \tilde{K}_A (and respectively \hat{K}_C, \hat{K}_A) according to

$$\begin{aligned}\tilde{H} &= \tilde{K}_C \oplus \tilde{K}_A \\ \hat{H} &= \hat{K}_C \oplus \hat{K}_A\end{aligned}\tag{32}$$

They are used to express the $(8+0)$, and respectively $(0+8)$, charge-conjugation matrices $\tilde{\mathcal{C}}$ ($\hat{\mathcal{C}}$) in terms of the $(4+4)$ charge-conjugation matrix \mathcal{C} via the similarity transformation

$$\tilde{\mathcal{C}} = \tilde{H} \cdot \mathcal{C} \cdot \tilde{H}^T\tag{33}$$

and the corresponding equation obtained by replacing “ \sim ” with “ \wedge ”.

For what concerns the Clifford’s Γ matrices in the $(4+4)$ -signature, they are mapped into the $(8+0)$ $\tilde{\Gamma}$ ’s according to

$$\Gamma^m \mapsto \tilde{\Gamma}^{\tilde{m}} = (\tilde{H}^T)^{-1} \cdot \Gamma^m \cdot \tilde{H}^T (\tilde{K}_V^T)_m^{\tilde{m}}\tag{34}$$

As usual, an analogous relation maps them into the $(0+8)$ -signature $\hat{\Gamma}$ ’s.

It is furthermore convenient to introduce the set of K_\star matrices, defined through

$$K_\star = \tilde{K}_\star \cdot \hat{K}_\star,\tag{35}$$

connecting the $(8+0)$ with the $(0+8)$ signatures.

Let us denote with \mathcal{W} (and respectively \mathcal{Z}) the transformations mapping the different signatures set of data according to the following symbolic actions:

$$\begin{aligned}\mathcal{W} &: (4+4) \leftrightarrow (4+4), & (8+0) &\leftrightarrow (0+8) \\ \mathcal{Z} &: (4+4) \leftrightarrow (8+0), & (0+8) &\leftrightarrow (0+8)\end{aligned}\tag{36}$$

Such transformations are explicitly realized by 24-dimensional matrices W_\star (and respectively Z_\star) acting on column vectors of the kind $(\phi_\star \ \tilde{\phi}_\star \ \hat{\phi}_\star)$. They are expressed in terms of the 8-dimensional block matrices K_\star, \tilde{K}_\star . Indeed we can put them into the form

$$\begin{aligned} W_\star &= \begin{pmatrix} \mathbf{1}_8 & 0 & 0 \\ 0 & 0 & K_\star \\ 0 & K_\star & 0 \end{pmatrix} \\ Z_\star &= \begin{pmatrix} 0 & \tilde{K}_\star & 0 \\ \tilde{K}_\star & 0 & 0 \\ 0 & 0 & \mathbf{1}_8 \end{pmatrix} \end{aligned} \quad (37)$$

The matrices \tilde{K}_\star, K_\star can be carefully chosen in such a way that the \mathcal{W}, \mathcal{Z} transformations can be regarded as the generators of the S_3 group (i.e. the signature triality group). This implies of course that the set of relations

$$\begin{aligned} \mathcal{W}^2 = \mathcal{Z}^2 &= \mathbf{1} \\ (\mathcal{W} \cdot \mathcal{Z})^3 &= \mathbf{1} \end{aligned} \quad (38)$$

must be satisfied.

We mention here that when an odd number of signatures is flipped, as it happens for the standard Wick rotation from the Minkowski into the Euclidean space, the \mathbf{Z}_2 group can not be realized with an action on the Γ 's matrices. Indeed when, let's say, the $m = 1$ direction is flipped, the corresponding Clifford matrix Γ^1 is mapped into $i\Gamma^1$, which leads to a \mathbf{Z}_4 group. A necessary condition to realize a \mathbf{Z}_2 group on Clifford's Γ matrices is that an *even* number of signatures has to be flipped (as it happens when changing e.g. $(++) \mapsto (--)$). This is due to the fact that the $\sigma_y = -ie_{12} + ie_{21}$ Pauli matrix, satisfying

$$\sigma_y^2 = \mathbf{1} \quad (39)$$

can be employed, allowing via a similarity transformation, to switch the sign

$$\sigma_y \cdot \mathbf{1}_2 \cdot \sigma_y^T = -\mathbf{1}_2 \quad (40)$$

In the case here considered a consistent choice for K_\star, \tilde{K}_\star is given by

$$\begin{aligned} K_V &= \sigma_y \oplus \sigma_y \oplus \sigma_y \oplus \sigma_y \\ K_C &= \mathbf{1}_8 \\ K_A &= \sigma_y \oplus \sigma_y \oplus \sigma_y \oplus \sigma_y \\ \tilde{K}_V &= \mathbf{1}_4 \oplus \sigma_y \oplus \sigma_y \\ \tilde{K}_C &= \mathbf{1}_4 \oplus \sigma_y \oplus \sigma_y \\ \tilde{K}_A &= \sigma_y \oplus \sigma_y \oplus \mathbf{1}_4 \end{aligned} \quad (41)$$

It is a straightforward exercise to verify the consistency of the full set of relations (38) with the above choice of matrices.

For what concerns \hat{K}_\star , they are immediately read from (35).

The above construction shows, as promised, that theories involving 8-dimensional Majorana-Weyl spinors in different signatures can all be dually related in such a way to close the S_3 group of permutations.

7 An application: the brane-molecule scan.

We present in this section an application of the signature-dualities properties induced by triality. It concerns the so-called brane-molecule scan of ref. [4], originally appeared in the Blencowe-Duff paper in NPB, and later revisited in the Duff's Tasi Lecture Notes. In these works the conditions for the existence of classical supersymmetric branes of arbitrary signatures embedded in flat target spaces whose signatures too are left arbitrary, are analyzed in details. Such conditions involve of course the property of spinors (Majorana, Weyl or Majorana-Weyl), plus extra identities involving the Clifford's Γ -matrices, which are needed in order to implement the kappa-symmetry.

The general result being presented in [4], here we just recall the allowed branes for $d = 10, 11, 12$ dimensional targets. We have

- i)* the superstrings $(1+1) \mapsto \{(1+9), (5+5) \text{ or } (9+1)\}$,
- ii)* the five-branes $\{(1+5) \mapsto \{(1+9) \text{ or } (5+5)\}$, as well as the mirror copies $(5+1) \mapsto \{(9+1) \text{ or } (5+5)\}$, obtained by exchanging space with time,
- iii)* the membranes $(1+2) \mapsto \{(9+2), (6+5), (5+6), (2+9), \text{ or } (1+10)\}$, as well as the mirror copies obtained by space-time exchange,
- iv)* finally, the non-minkowskian $(2+2) \mapsto \{(2+10), (6+6) \text{ or } (10+2)\}$ branes.

In all the above cases the target space-times can be recovered from the eight-dimensional spacetimes, according to the discussion in section 3. It is therefore clear that the “translation rules” expressing the various versions of the above theories in different signatures can be expressed in terms of the previous section results. In particular for the superstring, as well as for the $(2+2)$ brane, the three different versions are dually mapped in such a way to close the S_3 group.

In the original papers it was remarked the presence of a trivial \mathbf{Z}_2 symmetry obtained by space-time exchange, connecting e.g. the $(1+9)$ to the $(9+1)$ target spacetimes solutions of the superstring. However, the presence of the extra solution $(5+5)$ was left “unexplained”. It turns out to be connected with, let's say, the $(1+9)$ solution via another, non-trivial, generator of the S_3 triality group.

The above triality property holds for other classes of supersymmetric theories, like the 10-dimensional superYang-Mills theories, which are allowed in different signatures (as before in the $(1+9), (5+5), (9+1)$ cases).

8 Triality as an invariance.

In this section we wish to discuss another possible application of triality related to the Duff's viewpoint that a fundamental theory of everything should explain not only the dimensionality, but as well the signature of the spacetime. According to the Hull and Hull-Khuri results [5], the different versions in each given signature of a given theory are dually mapped one into another and therefore all equivalent. This is also the main content of the previous sections analysis,

where we pointed out the role played by the S_3 triality group in such a context.

However our own results admit another possible interpretation. Having identified S_3 as the duality group, a question that can be naturally raised is whether this group can be assumed as a group of invariance, providing a signature-independent framework for the description of our theories. All this is much in the same spirit as general relativity providing a coordinate independent scheme.

Such a question deserves of course a careful investigation. It should be mentioned that it is not so difficult to realize an S_3 -triality invariant formulation for strings, branes, etc. Other questions however, like the possibility that the S_3 group could be spontaneously broken, have no answer at present and need further investigation. A possible mechanism for producing a spontaneous breaking could use potential terms induced by the trilinear term given in formula (19).

The most natural setting to investigate such questions seems to be the eight-dimensional light-cone formulations of superstrings and branes, since the triality can be manifestly realized in this dimensionality.

We conclude this section pointing out the property that triality induced by supersymmetry strongly constraints the number of finite groups allowing a space-versus-time exchange symmetry. In bosonic theories, since all signatures are consistent and therefore allowed, the number of such groups is huge. In the supersymmetric case things are different. Let us discuss the 10-dimensional superstring theories to be definite. The allowed signatures of the target spacetimes are $(1 + 9)$, $(5 + 5)$, $(9 + 1)$. If a space-versus-time coordinate exchange symmetry is required, as a consequence only three groups arise, namely

- i)* the identity group $\mathbf{1}$, corresponding to a theory formulated in the single spacetime $(5 + 5)$,
- ii)* the \mathbf{Z}_2 group of symmetry, for a theory formulated by using two spacetimes copies $(1 + 9)$, $(9 + 1)$,
- iii)* finally, the S_3 group, which underlines a unified “space-time” theory requiring the whole set of 10-dimensional Majorana-Weyl spacetimes $(1 + 9)$, $(5 + 5)$, $(9 + 1)$.

9 Conclusions.

In this paper we have investigated various consequences of the triality automorphisms of $Spin(8)$. We have discussed in a suitable Majorana-Weyl basis (and partially extended) the original Cartan’s results [7] concerning the transformation properties of 8-dimensional vectors, chiral and antichiral spinors. Moreover, we have shown how triality affects the representation properties of the Clifford Γ matrices (the so-called Majorana-type representations). We have pointed out that triality naturally encodes the property that supersymmetry in higher-dimensions are consistently formulated in different-signature spacetimes. The S_3 triality group quite naturally arises in such a context. Indeed, we have been able to prove that different signature-versions of a given supersymmetric theory such, to give an example, as superstrings, can be dually transformed one into another with transformations, conveniently chosen, which allow closing the S_3 group.

The possible role of S_3 as a symmetry group for a space-versus-time exchange invariance has also been mentioned.

One of the main motivations to investigate such properties concerned the still highly speculative but quite fascinating topics that a supersymmetric fundamental theory could shed light on the nature of the time. This point of view has been advocated by Duff [4] in a series of papers and recently gained increased attention due to the results of Hull and collaborators [5].

The fact that supersymmetric theories seemingly related with the supposed F -theory find a natural formulation in a two-time physical world poses of course a strong challenge. Bars and collaborators [3], e.g., are exploring the possibility that the minkowskian time arises as a gauge-fixing of such a two-time physical world.

The present paper fits into this line of research. It is worth stressing however that, no matter which was the original motivation, the present paper contains a detailed account of mathematical results and property of triality which can be possibly used in technical analysis in different and more down-to-earth contexts. Just to mention an example which is currently under investigation, the here presented “translation rules” among different signatures can be used to define, let’s say, superYang-Mills theories in a $(5 + 5)$ signature. Such theories can be dimensionally reduced to an AdS $(++--)$ signature. Standard minkowskian 10-dimensional superYang-Mills theories do not admit of course such a dimensional reduction to AdS.

It is furthermore worth mentioning that some of the mathematical results here presented seems new, we have been unable to find them in the existing mathematical literature, at least not in such an explicit form as here given.

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Appendix 1: The $(0_S + 6_A)$ -representation for the 6-dimensional Γ ’s.

For completeness we furnish here a representation of the 6-dimensional Clifford’s Γ -matrices realized in terms of antisymmetric Γ ’s only. As discussed in the text, this realization allows to construct all Majorana-type representations for Clifford’s Γ -matrices up to dimension $d = 12$ with the help of the recursive formula (11).

The representation given below is the correct one to be used in order to express a 6-dimensional Euclidean space in the Majorana basis, which is consistent for $t = 6$, $s = 0$ when $\eta = -1$, as well as for $t = 0$, $s = 6$ and $\eta = +1$.

We constructed the $(0_S + 6_A)$ -representation out of the $(3_S + 3_A)$ -representation (which is easily obtained through (11)) after computing in such a case, for one of the above Euclidean spaces, the value of the \mathcal{B} -matrix and later finding a transformation U mapping $\mathcal{B} \mapsto U^* \mathcal{B} U^\dagger = \mathbf{1}$. As a consequence we obtain $\Gamma^i_{(0_S+6_A)} = U \Gamma^i_{(3_S+3_A)} U^\dagger$ according to (3). The extra Γ^7 matrix is also antisymmetric.

The final result, here presented for all Γ 's real (i.e. $\Gamma^{i^2} = -\mathbf{1}$), is given by:

$$\begin{aligned}
\Gamma^1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \Gamma^2 &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \\
\Gamma^3 &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} & \Gamma^4 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\Gamma^5 &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} & \Gamma^6 &= \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\Gamma^7 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{42}$$

Appendix 2:

The $(4_S + 4_A)$ -representation for the 8-dimensional Γ 's.

In the following three appendices we explicitly furnish the three Majorana-Weyl representations for the 8-dimensional Γ -matrices. Moreover in each case a specific realization of the \mathcal{P} , \mathcal{R} generators of the S_3 permutation group, leaving invariant the trilinear term (19) for the choice $\eta = -1$, i.e. for *commuting* spinors, is given (in the opposite case, $\eta = +1$, the trilinear term is

automatically vanishing for commuting spinors).

The Γ 's, as well as the generators \mathcal{P}, \mathcal{R} , are presented in real form (confront the discussion in section 5).

Each one of the three representations, being MW, admits a Γ^9 matrix of the kind

$$\Gamma^9 = \begin{pmatrix} \mathbf{1}_8 & 0 \\ 0 & -\mathbf{1}_8 \end{pmatrix}$$

while the Γ^i for $i = 1, 2, \dots, 8$ are decomposed according to

$$\Gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \tilde{\sigma}^i & 0 \end{pmatrix} \quad (43)$$

We point out that the 8-dimensional Γ -matrices introduced in this one and the two following appendices are not directly obtainable through the recursion formula (11). Rather, they have been conveniently chosen in order to provide a diagonal metric for both chiral and antichiral spinors which coincides, up to an overall sign, with the given diagonal metric for vectors.

In this appendix we present the results for the $(4_S + 4_A)$ -representation, the one which has to be used in order to introduce MW-spinors in a $t = 4, s = 4$ spacetime.

We have

$$\begin{aligned} \sigma^1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \sigma^2 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \sigma^3 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \sigma^4 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \sigma^5 &= \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} & \sigma^6 &= \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \end{aligned}$$

$$\sigma^7 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \sigma^8 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

while $\tilde{\sigma}_i = -\sigma_i^T$ for $i = 1, 2, 3, 4$ and $\tilde{\sigma}_i = \sigma_i^T$ for $i = 5, 6, 7, 8$, (i.e. the first four Γ -matrices are antisymmetric, the last four ones symmetric).

The diagonal charge-conjugation matrices are given by

$$\begin{aligned} C^{-1} &= \mathbf{1}_4 \oplus -\mathbf{1}_4 \\ \tilde{C}^{-1} &= -\eta \cdot C^{-1} \end{aligned} \tag{44}$$

A convenient basis for \mathcal{P}, \mathcal{R} is

$$\begin{aligned} \mathcal{P}_{V \mapsto V} \equiv \mathcal{P}_1 &= \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ \mathcal{P}_{A \mapsto C} \equiv \mathcal{P}_2 &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\ \mathcal{P}_{C \mapsto A} \equiv \mathcal{P}_3 &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{45}$$

and

$$\begin{aligned}
\mathcal{R}_{C \mapsto V} \equiv \mathcal{R}_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\mathcal{R}_{V \mapsto C} \equiv \mathcal{R}_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\mathcal{R}_{A \mapsto A} \equiv \mathcal{R}_3 &= \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}
\end{aligned} \tag{46}$$

Appendix 3:

The $(0_S + 8_A)$ -representation for the 8-dimensional Γ 's.

This representation is the one to be used in order to introduce a MW-basis for an Euclidean $t = 8$, $s = 0$ space when $\eta = -1$. In this case all Γ 's have to be assumed imaginary (in the following formulas we drop, as usual, the i).

The following formulas are directly obtainable from those presented in appendix 2 after applying a special case of the transformation (3) discussed in section 2. They are here presented

for completeness. We have

$$\begin{aligned}
\sigma^1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \sigma^2 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\sigma^3 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \sigma^4 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\sigma^5 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} & \sigma^6 &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \\
\sigma^7 &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} & \sigma^8 &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

Since the Γ^i are antisymmetric it follows that $\tilde{\sigma}^i = -\sigma^{iT}$ for any $i = 1, 2, \dots, 8$.

The two diagonal charge-conjugation matrices are

$$\begin{aligned}
C^{-1} &= \mathbf{1}_8 \\
\tilde{C}^{-1} &= \eta \cdot \mathbf{1}_8
\end{aligned} \tag{47}$$

A real basis for \mathcal{P}, \mathcal{R} is obtained by transforming the corresponding mappings in the $(4_S + 4_A)$

case. It is given by

$$\begin{aligned}
\mathcal{P}_{V \mapsto V} \equiv \mathcal{P}_1 &= \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \\
\mathcal{P}_{A \mapsto C} \equiv \mathcal{P}_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \\
\mathcal{P}_{C \mapsto A} \equiv \mathcal{P}_3 &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{48}$$

and

$$\begin{aligned}
\mathcal{R}_{C \mapsto V} \equiv \mathcal{R}_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\mathcal{R}_{V \mapsto C} \equiv \mathcal{R}_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

$$\mathcal{R}_{A \rightarrow A} \equiv \mathcal{R}_3 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad (49)$$

Appendix 4:

The $(8_S + 0_A)$ -representation for the 8-dimensional Γ 's.

This representation is the one to be used in order to introduce a MW-basis for an Euclidean $t = 0$, $s = 8$ space when $\eta = -1$. In this case as well as all Γ 's have to be assumed imaginary (in the following formulas we drop, as usual, the i).

As before the following formulas are obtained after transforming the corresponding formulas in the $(4_S + 4_A)$ case. We have

$$\begin{aligned} \sigma^1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \sigma^2 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \sigma^3 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \sigma^4 &= \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \sigma^5 &= \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} & \sigma^6 &= \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

$$\sigma^7 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \quad \sigma^8 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

while now $\tilde{\sigma}^i = \sigma^{iT}$ for any $i = 1, 2, \dots, 8$.

The two diagonal charge conjugation matrices are given by

$$\begin{aligned} C^{-1} &= \mathbf{1}_8 \\ \tilde{C}^{-1} &= -\eta \cdot \mathbf{1}_8 \end{aligned} \tag{50}$$

A real basis for \mathcal{P}, \mathcal{R} is given by

$$\begin{aligned} \mathcal{P}_{V \mapsto V} \equiv \mathcal{P}_1 &= \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ \mathcal{P}_{A \mapsto C} \equiv \mathcal{P}_2 &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\ \mathcal{P}_{C \mapsto A} \equiv \mathcal{P}_3 &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{51}$$

and

$$\begin{aligned}
\mathcal{R}_{C \mapsto V} \equiv \mathcal{R}_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\mathcal{R}_{V \mapsto C} \equiv \mathcal{R}_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\mathcal{R}_{A \mapsto A} \equiv \mathcal{R}_3 &= \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}
\end{aligned} \tag{52}$$

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